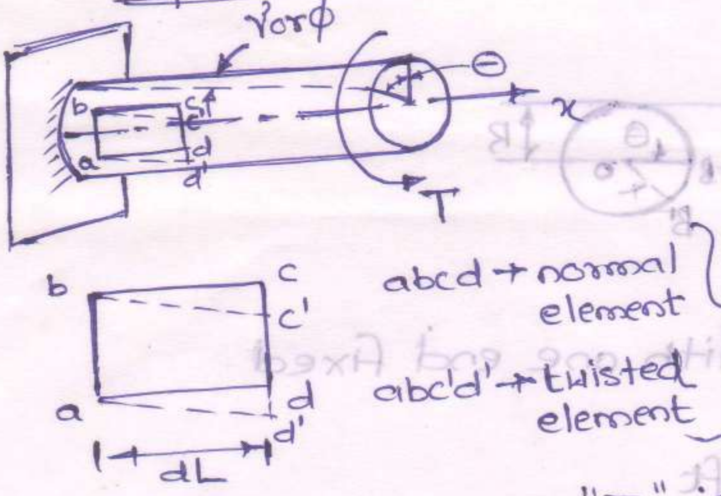
MODULE 4 :- TORSION

A member is said to be in torsion when it is subjected to moment about its axis [Note :- member ⇒ shaft i.e a circular cross-sectional straight bar used for power transmission]. The effect of torsional moment [also called twisting moment, torque and twisting couple] is to twist the member.

A shaft subjected to torsion and not accompanied by any other force such as bending or axial force is called said to be subjected to pure torsion.

Theory of Pure Torsion



abcd → normal element
abc'd' → twisted element

Consider a shaft of circular c/s and length "L". Let one end is fixed rigidly.

(a) All longitudinal lines are parallel to the axis.

(b) An element on the disc is rectangular i.e abcd.

If a twisting moment "T" is applied on the free end.

(a) The longitudinal lines are twisted by an angle "γ" or "ϕ"

(b) The rectangular element abcd becomes parallelogram abc'd'.

(1) Shear strain :- The longitudinal lines are rotated by an angle "ϕ" w.r.t fixed end. This angle is known as shear strain.

(2) Angle of twist :- The angle through which circular section at free end rotates about the longitudinal axis w.r.t the fixed end. Indicated by "θ".

Assumptions in the theory of pure torsion

(1) The material is homogeneous and isotropic

(2) The stresses are within the elastic limit i.e $\tau \propto \phi$.

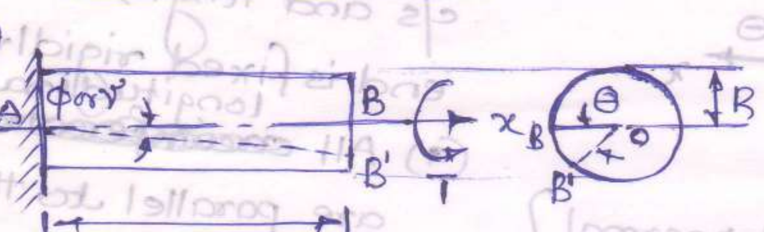
(3) Cross-sections which are plane before applying twisting moment remain plane even after the application of twisting moment i.e no warping occurs.

(4) Radial lines remain radial even after torque is applied.

(5) The twist along the shaft is uniform.

Torsional Equation

(a)



Consider a shaft with one end fixed.

Let,

L = length of shaft.

R = radius of shaft.

T = torque applied at free end.

AB = Line on shaft s/c parallel to axis.

B = point on the surface with center "O".

When torque "T" is applied, the shaft twists i.e point "B" rotates to point "B'" and "AB" rotates to "AB'" longitudinally about the axis.

$\therefore$ shear strain, $\phi = \frac{\tau_s}{G}$ $\longrightarrow$ (a)

Arc length, $BB' = R\theta = L\phi$ $\longrightarrow$ (b) Substituting from (a).

$\therefore R\theta = L\frac{\tau_s}{G}$

$$\therefore \boxed{\frac{\tau_s}{R} = \frac{G\theta}{L}} \quad \text{— (1)}$$

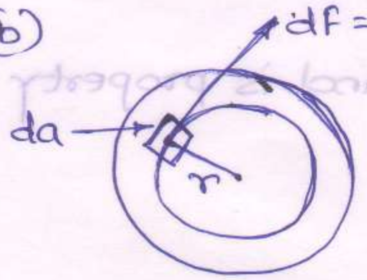
τ_s = shear stress at s/c
G = modulus of rigidity.

If point "B" is considered at any distance "r" from center, then,

$$\frac{\tau_s}{R} = \frac{\tau_r}{r} = \frac{G\theta}{L} \quad \text{— (c)}$$

Hence shear stress increases linearly from zero at axis to maximum (τ_s) at surface.

(b)



Now, let
da = elemental area at a distance "r" from the axis.

The resisting force, $dF = \tau\, da$ — (d)

Resisting torsional moment, $dT = dF \times r$
$= \tau\, da \times r$ — (e)

from eq^n (c), $\frac{\tau_s}{R} = \frac{\tau}{r}$ substitute in (e)

$$\therefore dT = \frac{\tau_s}{R} r^2 da \quad \text{— (f)}$$

Total resisting torque, $T = \frac{\tau_s}{R} \Sigma r^2 da$

$$\therefore \boxed{T = \frac{\tau_s}{R} J} \quad \text{— (2)}$$

where, $J = \Sigma r^2 da$, polar moment of inertia of section

$\therefore$ From equation (1) and (2).

$\therefore$ Torsion eq^n, $\frac{T}{J} = \frac{\tau_s}{R} = \frac{G\theta}{L}$ or $\boxed{\frac{T}{J} = \frac{\tau}{r} = \frac{G\theta}{L}}$ — (3)

where, r = distance of element from shaft Center. in mm

T = torque in Nmm
J = polar moment of inertia in mm^4
τ = shear stress at "r" in N/mm^2
G = modulus of rigidity in N/mm^2
θ = angle of twist in radians

L = shaft length in mm

Polar Modulus

Torsion equation, $\frac{T}{J} = \frac{\tau}{r}$

at $r = R$ we have $\tau = \tau_s$.

$$\therefore \frac{T}{J} = \frac{\tau_s}{R}$$

$$\therefore T = \frac{J}{R}\tau_s$$

$$\therefore \boxed{T = Z_p \tau_s} \quad \text{— (1)}$$

where, Z_p = Polar modulus of section and is property of the section.

(a) Solid shaft:

$$J = \frac{\pi}{32}d^4 \quad \text{and} \quad R = \frac{d}{2}$$

$$\therefore Z_p = \frac{J}{R} = \frac{\pi}{16}d^3 \quad \text{— (2)}$$

(b) Hollow shaft

$$J = \frac{\pi}{32}\left(d_o^4 - d_i^4\right) \quad \text{and} \quad R = \frac{d_o}{2}.$$

$$\therefore Z_p = \frac{\pi}{16}\frac{\left(d_o^4 - d_i^4\right)}{d_o} \quad \text{— (3)}$$

Power transmitted.

Consider a shaft rotating at "N" r.p.m and subjected to torque "T"

$$\therefore \text{Power, } P = T.\omega \quad \text{— (a)}$$

angular velocity, $\omega = \frac{2\pi N}{60}$ rad/s

$$\therefore P = \frac{2\pi NT}{60} \text{ (Watts) W}$$

$$\therefore P = \frac{2\pi NT}{60 \times 1000} \text{ KW} \quad \text{— (b)}$$

In metric units power is in Horse Power

$$1 HP = 0.745 \text{ KW} \quad \text{— (c)}$$

Definitions :-

(1) Torsional strength :- "Torque per unit maximum shear stress". It is also known as the efficiency of a shaft.

$$\therefore \text{ Torsional strength } = \frac{T}{\tau_s} \quad —(a).$$

From torsion equation, $\frac{T}{J} = \frac{\tau_s}{R} = \frac{G\theta}{L}$.

$$\text{but, } \frac{T}{\tau_s} = \frac{J}{R} = Z_p \quad —(b)$$

Hence torsional strength can be represented by polar modulus of section.

(2) Torsional Rigidity :- "Torque required to produce a unit angle of twist in a specified length of the shaft".

$$\therefore \text{ Torsional rigidity / Torsional stiffness } = \frac{T}{\theta} \quad —(c) = \frac{GJ}{L}$$

(3) Torsional flexibility :- "Angle of twist per unit torque". Reciprocal of torsional rigidity.

$$\therefore \text{ Torsional flexibility } = \frac{\theta}{T} \quad —(d) = \frac{L}{GJ}.$$

Problems

(1) Calculate the maximum intensity of shear stress induced and the angle of twist produced in degress in solid shaft of 100 mm diameter, 10 m long, transmitting 112.5 kW at 150 rpm. Take $G = 82\ kN/mm^2$.

Soln :- Data Given

$d = 100\ mm$

$L = 10\ m = 10 \times 10^3\ mm$

$P = 112.5\ kW = 112.5 \times 10^3\ Watts = 112.5 \times 10^6\ Nmm/s$

$N = 150\ rpm$.

$G = 82\ kN/mm^2 = 82 \times 10^3\ N/mm^2$

5/4

From, torsion equation, $\frac{T}{J} = \frac{\tau}{r} = \frac{G\theta}{L}$ — (a)

(a) Power transmitted, $P = \frac{2\pi NT}{60}$

$\therefore$ Torque, $T = \frac{60P}{2\pi N}$

$\therefore T = \frac{60 \times 112.5 \times 10^6}{2\pi \times 150}$

$\therefore \boxed{T = 7.16 \times 10^6}$ Nmm — (1)

(b) Polar moment of Inertia, $J = \frac{\pi d^4}{32}$

$J = \frac{\pi\, 100^4}{32}$

$\therefore \boxed{J = 9.817 \times 10^6}$ mm^4 — (2)

(c) Max. intensity of shear stress occurs at outer fiber.

$\therefore \frac{T}{J} = \frac{\tau_{max}}{R}$

$\therefore \tau_{max} = \frac{7.16 \times 10^6}{9.817 \times 10^6} \times \frac{100}{2}$

$\therefore \boxed{\tau_{max} = 36.467}$ N/mm^2

(d) Angle of twist,

$\frac{T}{J} = \frac{G\theta}{L}$

$\therefore \theta = \frac{7.16 \times 10^6}{9.817 \times 10^6} \times \frac{10 \times 10^3}{82 \times 10^3}$

$\boxed{\theta = 88.94 \times 10^{-3}}$ rad.

Note:- π rad $= 180^\circ$

$\therefore 88.94 \times 10^{-3}$ rad $=$

$\therefore \theta = 88.94 \times 10^{-3} \times \frac{180}{\pi} = \underline{5.096^\circ}$.

(2) A hollow steel shaft transmits 200 kW of power at 150 rpm. The total angle of twist in a length of 5 m of shaft is 3°. Find the inner and outer diameters of the shaft if the permissible shear stress is 60 MPa. Take, G = 80 GPa.

Soln :- Data Given

$P = 200\,kW = 200\times10^3\ W$

$N = 150$ rpm.

$L = 5$ m

$\theta = 3^\circ = 3\times\frac{\pi}{180} = 0.0523$ rad.

$\tau_{permissible} = 60\,MPa = 60\times10^6\ N/m^2$

$G = 80\,GPa = 80\times10^3\ N/mm^2 = 80\times10^9\ N/m^2$

(a) From Torsion equation, $\frac{T}{J} = \frac{\tau}{r} = \frac{G\theta}{L}$ — (a)

Power transmitted, $P = \frac{2\pi NT}{60}$ W

$\therefore$ Torque, $T = \frac{200\times10^3\times60}{2\pi\times150}$

$\therefore \boxed{T = 12.732\times10^3}$ N m — (1)

(b) From eqn (a), $\frac{T}{J} = \frac{G\theta}{L}$

$$\therefore \frac{12.732\times10^3}{\frac{\pi}{32}\left[d_o^4-d_i^4\right]} = \frac{80\times10^9\times0.0523}{5}$$

$$\therefore \left(d_o^4-d_i^4\right) = \frac{12.732\times10^3\times32}{\pi\times8368\times10^6}$$

$$\therefore \left(d_o^4-d_i^4\right) = 0.1549\times10^{-3} \quad (2)$$

(c) From eqⁿ (a), $\frac{\tau_{per}}{R_o} = \frac{G\theta}{L}$ $\left(\because \tau_{per} = \tau_{max}\right)$

$\therefore R_o = \frac{60\times10^6 \times 5}{80\times10^9 \times 0.0523}$

$\therefore \boxed{R_o = 0.0717\ m}$ — (3)

(d) Hence outer diameter, $d_o = 2R_o = 0.1434\,m$

$\therefore \boxed{d_o = 143.4\ mm}$ — (4)

Substituting (4) in eqⁿ (2).

$\left(d_o^4 - d_i^4\right) = 0.1594\times10^{-3}$

$\therefore d_i^4 = (0.1434)^4 - \left(0.1594\times10^{-3}\right)$

$\therefore d_i^4 = 2.6345\times10^{-4}$

$\therefore d_i = 0.1274\ m$

$\therefore$ Inner diameter, $\boxed{d_i = 127.4\ mm}$ — (5)

(3) Determine the diameter of solid shaft which will transmit 440 KW at 280 rpm. The angle of twist must not exceed one degree per meter length and the maximum torsional shear stress is to be limited to $40\,N/mm^2$. Assume $G = 84\,kN/mm^2$.

Solⁿ :- Data :-

$P = 440\,KW = 440\times10^6\ Nmm/s$.

$G = 84\,kN/mm^2 = 84\times10^3\ N/mm^2$

$N = 280$ rpm.

$\theta = 1^\circ = \frac{\pi}{180}$ rad.

$L = 1m = 1000\,mm$

$\tau_{max} = 40\,N/mm^2$

∴ From, Torsional equation, $\frac{T}{J} = \frac{\tau}{r} = \frac{G\theta}{L}$ — (1)

(a) We have, $\frac{T}{J} = \frac{\tau}{r}$ Since $\tau = \tau_{max}$, $r = R$.

Power transmitted, $P = \frac{2\pi NT}{60}$

$$440\times10^6 = \frac{2\pi \times 280 \times T}{60}$$

∴ Torque, $\boxed{T = 15006037.49}$ N mm —(a)

∴ Polar modulus, $J = \frac{\pi d^4}{32}$ —(b)

Shaft radius, $R = \frac{d}{2}$ —(c)

Put (a), (b) and (c) in equation (1)

$$\therefore \frac{15006037.49}{\left(\frac{\pi d^4}{32}\right)} = \frac{40}{\left(\frac{d}{2}\right)}$$

$\therefore d^3 = 1910628.034$

$\therefore \boxed{d = 124.086}$ mm —(I)

(b) Considering angle of twist, $\theta \leq 1°$, from eq^n (1)

$$\frac{T}{J} = \frac{\tau_{max}}{R} = \frac{G\theta}{L}$$

[Note:- Max shaft dia obtained from the combination of eq^n (1) is the min value of shaft dia req. If d=101 mm is selected shaft will fail as it requires, d=124 mm to withstand τ_{max}.]

$$\frac{15006037.49}{\left(\frac{\pi d^4}{32}\right)} = \frac{84\times10^3 \times \pi/180}{1000}$$

$\therefore d^4 = 104263467.1$

$\therefore \boxed{d = 101.049 \text{ mm}}$ —(II)

Thus min dia $d = 124.086$ mm

(4) The working condition to be satisfied by a shaft transmitting power are (i) the shaft must not twist more than 1° in a length of 15 times diameter. (ii) the shear stress must not exceed 80 MN/m^2. What is the actual working stress and diameter of the shaft to transmit 736 KW at 200 rpm? Take shear modulus as 80 GN/m^2.

Soln: Data.

$\theta = 1^\circ = \pi/180$ rad

$L = 15\,d$

$\tau_{max} = 80\ MN/m^2 = 80\ N/mm^2$

$P = 736\ KW = 736 \times 10^6\ Nmm/s$

$N = 200$ rpm.

$G = 80\,GN/m^2 = 80 \times 10^3\ N/mm^2$

from torsional equation

$$\frac{T}{J} = \frac{\tau}{r} = \frac{G\theta}{L} \quad \text{— (i)}$$

(a) We have $\frac{T}{J} = \frac{G\theta}{L}$ — (a)

Polar moment of Inertia, $J = \frac{\pi d^4}{32}$ — (b).

Power transmitted, $P = \frac{2\pi N T}{60}$

$\therefore$ Torque transmitted, $T = \frac{736 \times 10^6 \times 60}{2\pi \times 200}$

$$\therefore \boxed{T = 35141411.43}\ Nmm \quad \text{— (c)}$$

Put (b) and (c) in (a)

$$\frac{35141411.43}{\left(\frac{\pi d^4}{32}\right)} = \frac{80 \times 10^3 \times (\pi/180)}{15\,d}$$

$\therefore d^3 = 3845414.92$

$$\therefore \boxed{d = 156.66}\ mm \quad \text{— (I)}$$

(b) From eqn (i), we have, $\frac{T}{J} = \frac{\tau}{r}$

Since, $\tau = \tau_{max}$, $r = R = \frac{d}{2}$

$$\therefore \frac{3514141.43}{\left(\frac{\pi d^4}{32}\right)} = \frac{80}{\frac{d}{2}}$$

$$\therefore d^3 = 2237171.735$$

$$\therefore \boxed{d = 130.78} \text{ mm} \quad \text{— (II)}$$

(c) Since diameter calculated under consideration for angle of twist is more than dia calculated for shear stress

i.e $d = 156.66 > 130.78$

Thus shaft diameter, $\boxed{d = 156.66}$ mm.

(d) The working stress $\tau = \frac{T}{J} r$

$$\therefore \tau = \frac{3514141.43}{\left(\frac{\pi d^4}{32}\right)} \left(\frac{d}{2}\right) \quad \text{Taking } d = 156.66 \text{ mm.}$$

$\therefore \tau = 46.549 \text{ N/mm}^2 < \tau_{max}$ Hence safe.

(5) During tests on a sample of steel bar 25 mm in diameter it is found that the pull of 50KN produces an extension of 0.095 mm on a length of 200 mm and a torque of 200 Nm produces an angular twist of 0.9 degrees on a length of 250 mm. Find the Poisson's Ratio of the steel.

Soln :- Data

$d = 25$ mm

$P = 50$ KN

$\Delta L_1 = 0.095$ mm

$L_1 = 200$ mm

$T = 200$ Nm $= 200 \times 10^3$ Nmm

$\theta = 0.9° = 0.9 \times \frac{\pi}{180}$

$L_2 = 250$ mm.

(a) Deformation, $\Delta L_1 = \frac{PL_1}{AE}$ — (a)

$\therefore$ Young's modulus, $E = \frac{PL_1}{\Delta L_1 A}$

$$\therefore E = \frac{50\times10^3 \times 200}{0.095 \times \frac{\pi}{4} 25^2}$$

$$\therefore \boxed{E = 21440.34}\ N/mm^2 \quad — (1)$$

(b) From torsional equation, $\frac{T}{J} = \frac{G\theta}{L}$ — (b)

$$\text{Modulus of rigidity, } G = \frac{200\times10^3 \times 250}{\frac{\pi}{32} 25^4 \times 0.9 \frac{\pi}{180}}$$

$$\therefore \boxed{G = 83002.314}\ N/mm^2 \quad — (2)$$

(c) Relation between E and G.

$$E = 2G(1+\nu) \quad — (c)$$

$$\therefore \nu = \frac{E}{2G} - 1 = \left(\frac{21440.34}{2\times83002.314}\right) - 1$$

$$\therefore \text{Poisson's Ratio, } \boxed{\nu = 0.2917} \quad — (3)$$

JUNE/JULY '14 (10 marks)

(6) Find the diameter of the shaft required to transmit 60KW at 150 rpm if the maximum torque is 25% more than the mean torque for a maximum shear stress of 60MPa. Find also the angle of twist in a length of 4m. Take G = 80GPa.

Soln:- Data

$P = 60KW = 60\times10^6$ Nmm/s	$L = 4m$
$N = 150$ rpm	$G = 80GPa$
$T_{max} = 25\% T$	
$\tau_{max} = 60MPa$	

Sol^n (a) Torsional equation, $\frac{T}{J} = \frac{\tau}{r} = \frac{G\theta}{L}$ — (1)

(b) Power transmitted, $P = \frac{2\pi N T}{60}$

$\therefore$ Torque, $T = \frac{60 \times 60 \times 10^6}{2\pi\, 150} = \underline{\underline{3819718.634}}$ Nmm — (a)

Polar moment of inertia, $J = \frac{\pi}{32} d^4$ — (b)

$\therefore$ From equation (1), $\frac{T}{J} = \left(\frac{\tau_{max}}{R}\right)$

for maximum torque, $\frac{T_{max}}{J} = \frac{\tau_{max}}{R}$ where, $T_{max} = 25\%$ T

$\therefore \frac{1.25 \times 3819718.634}{\frac{\pi}{32} d^4} = \frac{60}{\left(\frac{d}{2}\right)}$

$\therefore$ Shaft diameter, $\boxed{d = 74 \text{ mm}}$ — (2)

(c) From equation (1), $\frac{T_{max}}{J} = \frac{G\theta}{L}$

$\therefore \frac{1.25 \times 3819718.634}{\frac{\pi}{32} d^4} = \frac{80 \times 10^3 \times \theta}{4 \times 10^3}$

$\therefore$ Angle of twist, $\theta = 0.081$ rad

$\therefore \theta = 0.081 \times \frac{180}{\pi}$

$\therefore \boxed{\theta = 4.64^\circ}$

JUNE/JULY'15 (12 marks)

(7) A hollow shaft of diameter ratio 3/8 is required to transmit 588 kW at 110 rpm, the maximum torque being 20% more than mean. Shear stress is not to exceed 63 N/mm^2

and twist in a length of 3m not to exceed 1.4 degrees. Calculate external diameter of shaft which would satisfy these conditions. Take modulus of rigidity = 84 GPa.

Soln: Data:

$\frac{d_i}{d_o} = \frac{3}{8}$

Note:- { If T_{max} is 120% of mean torque then $T_{max} = \frac{120}{100} T = 1.2T$ }

$P = 588 kW = 588 \times 10^6$ Nmm/s

$N = 110$ rpm.

$T_{max} = \left(1 + \frac{120}{100}\right) T = 2.2 T$.

$\tau_{max} = 63 N/mm^2$

$\theta = 1.4^\circ$

$L = 3m = 3 \times 10^3$ mm

$G = 84$ GPa.

(a) Torsional equation, $\frac{T}{J} = \frac{\tau}{r} = \frac{G\theta}{L}$ — (a).

in this problem, $\frac{T_{max}}{J} = \frac{\tau_{max}}{R} = \frac{G\theta}{L}$ — (b)

(b) From equation (a) and (b) we have.

$\frac{T_{max}}{J} = \frac{\tau_{max}}{R}$ — (c)

Power transmitted, $P = \frac{2\pi N T}{60}$ W.

$\therefore$ Torque, $T = \frac{60 \times 588 \times 10^6}{2\pi \times 110} = \underline{51045330.84}$ N

$\therefore$ Max. Torque, $T_{max} = 2.2 T = \underline{112299727.8}$ Nm

Polar moment of inertia for hollow shaft, $J = \frac{\pi}{32}(d_o^4 - d_i^4)$

$\therefore J = \frac{\pi}{32}\left\{d_o^4 - \left(\frac{3}{8}d_o\right)^4\right\}$

$\therefore J = \frac{\pi}{32} \times 980.22 \times 10^{-3} d_o^4$ //

$\therefore$ From eq^n (c), $\dfrac{112299727.8}{\frac{\pi}{32}\times 980.22\times10^{-3}d_o^4}\left(\dfrac{d_o}{2}\right) = 63$

$\therefore$ Outer diameter of shaft, $\boxed{d_o = 210\ mm}$ — ①

(c) Check for outer diameter w.r.t angle of twist.

$$\frac{T_{max}}{J} = \frac{G\theta}{L}$$

$$\therefore \frac{112299727.8}{\frac{\pi}{32}\times 980.22\times10^{-3}d_o^4} = \frac{84\times10^3\times1.4\times\pi}{180\times3\times10^3}$$

$$\therefore d_o^4 = 1705657449$$

$\therefore$ outer diameter, $\boxed{d_o = 203.22\ mm}$ — ②

(d) Since shaft outer diameter w.r.t max shear stress is more than the outer diameter w.r.t angle of twist i.e $d_o = 210 > 203.22$.

Hence outer diameter, $d_o = 210\ mm$.

inner diameter, $d_i = \frac{3}{8}d_o = \frac{3}{8}\times 210$

$\therefore \boxed{d_i = 78.75\ mm}$ — ③

JUNE/JULY'13 (10 marks)

(8) A hollow circular steel shaft has to transmit 60KW at 210rpm such that the maximum shear stress does not exceed 60MPa. If the ratio of internal to external diameter is equal to 3/4 and the value of rigidity modulus is 84GPa. Find the dimensions of the shaft and the angle of twist in a length of 3m.

Soln:- Data:

$P = 60KW = 60\times10^6\ Nmm/s$

$N = 210\ rpm$

$\tau_{max} = 60MPa$.

$d_i/d_o = 3/4$

$G = 84GPa = 84\times10^3\ MPa$.

$L = 3m = 3\times10^3\ mm$

Torsional equation, $\frac{T}{J} = \frac{\tau}{r} = \frac{G\theta}{L}$ — (a)

(a) From eq^n (a) $\frac{T}{J} = \frac{\tau_{max}}{R}$ — (b)

Power transmitted, $P = \frac{2\pi N T}{60}$ W

$\therefore$ Torque, $T = \frac{60 \times 60 \times 10^6}{2\pi \times 210} = \underline{2728370.453}$ Nmm

Polar moment of inertia, $J = \frac{\pi}{32}(d_o^4 - d_i^4) = \frac{\pi}{32}\left\{\frac{(4d_o)^4 - (3d_o)^4}{(4)^4}\right\}$

$\therefore J = 0.067 d_o^4$

From eq^n (b), $\frac{2728370.453}{0.067 d_o^4} = \frac{60}{(d_o/2)}$

$\therefore$ Outer dia. $\boxed{d_o = 69.75}$ mm — (1)

Inner diameter, $d_i = \frac{3}{4} d_o = \underline{52.3125}$ mm — (2)

(b) From eq^n (a), $\frac{T}{J} = \frac{G\theta}{L}$

$\therefore \theta = \frac{2728370.453 \times 3 \times 10^3}{\frac{\pi}{32}[69.75^4 - 52.3125^4] \times 84 \times 10^3}$

$\therefore \theta = 0.06134$ rad.

$\therefore$ Angle of twist, $\boxed{\theta = 3.514^\circ}$ — (3)

Dec'14 (10 marks)

(a) A hollow shaft having an inside diameter 60% of its outer diameter, is to replace a solid shaft transmitting the same power at the same speed. Calculate the percentage saving in material, if the material to be used is also the same.

Solⁿ: Data:

$P_s = P_h = P$

$N_s = N_h = N$

$d_i = 0.6 d_o$

(a) Torsional equation, $\frac{T}{J} = \frac{\tau}{r} = \frac{G\theta}{L}$ — (a)

(b) Since power and speed for solid and hollow shaft are same from power transmitted formula.

$P = \frac{2\pi N T}{60}$ Watt, we have.

Torque, $T_s = T_h = T$ — (b)

Hence taking torsional strength to be same for both shaft

ie $\frac{T}{\tau_{max}}$ is same for solid and hollow shaft.

(c) From eqⁿ (a)

(i) For solid shaft: $\frac{T}{J} = \frac{\tau_{max}}{R}$

$\therefore \frac{T}{\tau_{max}} = \frac{\pi}{32} d^4 \Big/ \frac{d}{2} = \frac{\pi}{16} d^3$ — (c)

(ii) For hollow shaft:-

$\frac{T}{\tau_{max}} = \frac{\pi}{32}(d_o^4 - d_i^4) \Big/ (d_o/2) = \frac{\pi}{16} d_o^3 [1 - 0.6^4]$ — (d)

Comparing (c) and (d)

$\frac{\pi}{16} d^3 = \frac{\pi}{16} 870 \times 10^{-3} d_o^3$

$\therefore \boxed{d = 0.954 d_o}$ — (e)

(d) % material saved on replacing solid shaft with hollow shaft $= \frac{W_s - W_h}{W_s} \times 100$

{ Weight of solid shaft, $W_s = (\rho g L A)_s$
Weight of hollow shaft, $W_h = (\rho g L A)_h$ } Same material $\rho_s = \rho_h = \rho$ and $L_s = L_h = L$.

$\therefore$ % material saved = $\dfrac{\frac{\pi}{4}d^2 - \frac{\pi}{4}(d_o^2 - d_i^2)}{\frac{\pi}{4}d^2} \times 100$

$= \dfrac{d^2 - (d_o^2 - 0.6^2 d_o^2)}{d^2} \times 100$

$= \left\{ 1 - \dfrac{d_o^2(1 - 0.6^2)}{0.954^2 d_o^2} \right\} \times 100$

$= 0.2967 \times 100$

$\therefore$ % material saved = 29.67 %. — (1)

STEPPED SHAFTS

Shafts with sections of different c/s in steps of small shafts are analysed by finding the torque restricted by each portion and then the individual effect is combined.

(1) At fixed end torque of required magnitude develops to keep the shaft in equilibrium.

(2) The torques developed at the ends of any portion are equal and opposite.

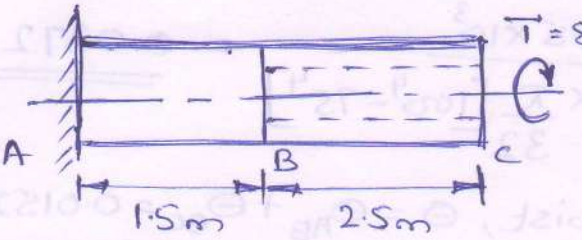
(3) At common point between two portion angle of twist is the same.

{Note :- ccw = +ve and cw = -ve from right to left.}

Problems :

(1) The shaft shown is securely fixed at "A" and is subjected to a torque of 8 KNm. If portion "AB" is solid shaft of 100 mm diameter and portion BC is hollow with external diameter 100 mm and internal diameter 75 mm, find the maximum stress and maximum angle of twist. Take, $G = 80\ kN/mm^2$

Soln:



Data:-

$T = 8\,kNm = 8\times10^6\,Nmm$

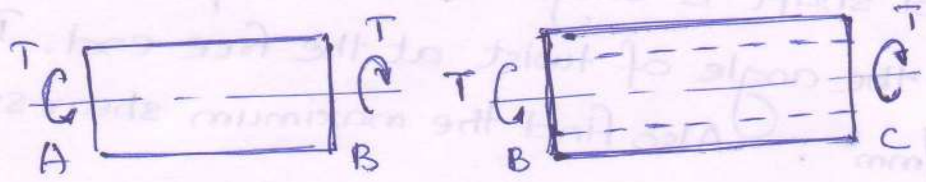
$d = 100\,mm$

$d_o = 100\,mm$

$d_i = 75\,mm$

$G = 80\,kN/mm^2$

(a) Free body diagram



(b) From torsional equation, $\frac{T}{J} = \frac{\tau}{r} = \frac{G\theta}{L}$ — (a)

(i) For AB :-

$$\frac{T}{\frac{\pi}{32}d^4} = \frac{\tau_{max}}{\left(\frac{d}{2}\right)} \quad \therefore \tau_{max} = \frac{8\times10^6\times16}{\pi\,100^3}$$

$$\therefore \boxed{\tau_{max} = 40.743}\,N/mm^2 \quad \text{— (1)}$$

(ii) For BC :-

$$\frac{T}{\frac{\pi}{32}(d_o^4-d_i^4)} = \frac{\tau_{max}}{(d_o/2)}$$

$$\therefore \tau_{max} = \frac{8\times10^6\times16\times100}{\pi\,(100^4-75^4)} = 59.6\,N/mm^2 \quad \text{— (2)}$$

From (1) and (2) $\tau_{max} = 59.6\,N/mm^2 > 40.7\,N/mm^2$

Hence shear stress is maximum in BC

$$\therefore \boxed{\tau_{max} = 59.6\,N/mm^2}$$

(c) From eqn (a), $\frac{T}{J} = \frac{G\theta}{L}$

(i) For AB :-

$$\theta_{AB} = \frac{8\times10^6\times1.5\times10^3}{\frac{\pi}{32}100^4\times80\times10^3} = 0.01527\,rad$$

$$\therefore \boxed{\theta_{AB} = -0.01527\,rad} \quad \text{— (3) CW rotation}$$

(ii) For BC : $\theta_{BC} = \frac{TL}{JG}$

$$\therefore \theta_{BC} = \frac{8\times10^6 \times 2.5\times10^3}{80\times10^3 \times \frac{\pi}{32}\left[100^4 - 75^4\right]} = -0.0372 \text{ rad } \{\text{c.w. rotation}$$

$\therefore$ Total angle of twist, $\theta = \theta_{AB} + \theta_{BC} = 0.01527 + 0.0372$

$$\therefore \boxed{\theta = 0.0524 \text{ rad}}$$

(2) A stepped shaft is subjected to torque as shown. Determine the angle of twist at the free end. Take $G = 80\,kN/mm^2$. Also find the maximum shear stress in any step.

Soln :-

Data :-

$d_{AB} = d_o = 100\,mm$

$d_i = 80\,mm$

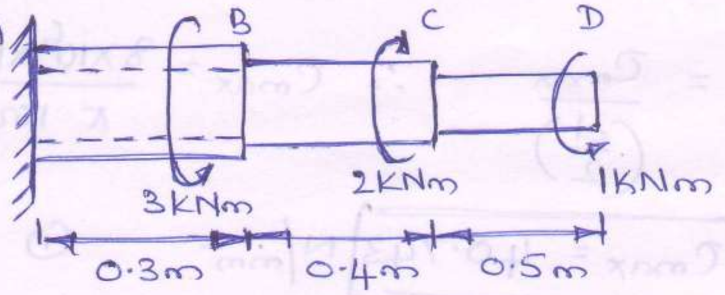
$d_{BC} = 80\,mm$

$d_{CD} = 60\,mm$.

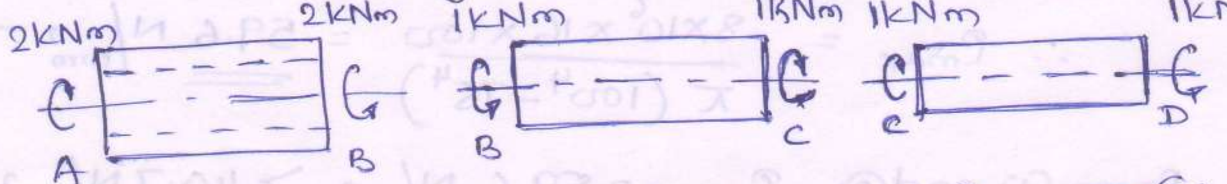
$L_{AB} = 0.3\times10^3\,mm$

$L_{BC} = 0.4\times10^3\,mm$

$L_{CD} = 0.5\times10^3\,mm$



(a) Free body diagram.



(b) Angle of twist, $\theta = \theta_{AB} + \theta_{BC} + \theta_{CD}$ —— (a)

$$\theta_{AB} = \frac{T_{AB} L_{AB}}{J_{AB} G} = \frac{2\times10^6 \times 0.3\times10^3}{\frac{\pi}{32}(100^4 - 80^4)\times 80\times10^3} = 0.00129 \text{ rad}$$

$$\theta_{BC} = \frac{T_{BC} L_{BC}}{J_{BC} G} = \frac{1\times10^6 \times 0.4\times10^3}{\frac{\pi}{32}\times 80^4 \times 80\times10^3} = -0.001243 \text{ rad}$$

$$\theta_{CD} = \frac{T_{CD} L_{CD}}{J_{CD} G} = \frac{1\times10^6 \times 0.5\times10^3}{\frac{\pi}{32}\times 60^4 \times 80\times10^3} = 0.00491 \text{ rad}$$

$\therefore \theta = 0.004957$ rad —— ①

(c) Shear stress, $\tau_{max} = \frac{R}{J} T$

$$(\tau_{max})_{AB} = \frac{(100/2)}{\frac{\pi}{32}[100^4 - 80^4]} \times 2\times10^6 = \underline{\underline{17.252 \ N/mm^2}}$$

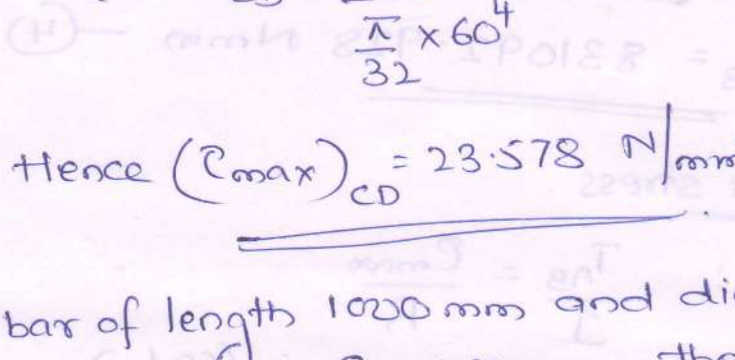
$$(\tau_{max})_{BC} = \frac{(80/2)}{\frac{\pi}{32} 80^4} \times 1\times10^6 = \underline{\underline{9.947 \ N/mm^2}}$$

$$(\tau_{max})_{CD} = \frac{(60/2)}{\frac{\pi}{32} \times 60^4} \times 1\times10^6 = \underline{\underline{23.578}} \ N/mm^2$$

Hence $\underline{\underline{(\tau_{max})_{CD} = 23.578 \ N/mm^2}}$

(3) A bar of length 1000 mm and diameter 60 mm is centrally bored for 400 mm, the bore diameter being 30 mm. If the two ends are fixed and is subjected to a torque of 2 kNm, find the maximum stresses developed in the two portions.

Soln :-



Data :-
$T = 2 \ kNm = 2\times10^6 \ Nmm$
$d = 60 \ mm$
$d_o = 60 \ mm$
$d_i = 30 \ mm$
$L_{AB} = 600 \ mm$
$L_{BC} = 400 \ mm$

(a) Torsional equation, $\frac{T}{J} = \frac{\tau}{r} = \frac{G\theta}{L}$ — (a)

(b) Net Torque, $T = T_{AB} + T_{BC}$ — (b)

$$\therefore \underline{\underline{T_{AB} + T_{BC} = 2\times10^6 \ Nmm}} \ — (1)$$

(c) Angle of twist, $\theta_{AB} = \theta_{BC} = \theta$. — (c)

$$\therefore \left(\frac{TL}{JG}\right)_{AB} = \left(\frac{TL}{JG}\right)_{BC} \ — (d)$$

Since material is same $G_{AB} = G_{BC}$, then eqⁿ (d)

$$\frac{T_{AB} \times 600}{\frac{\pi}{32} \times 60^4} = \frac{T_{BC} \times 400}{\frac{\pi}{32}\left[60^4 - 30^4\right]}$$

$$\therefore \boxed{T_{AB} = 0.711\ T_{BC}} \quad \text{--- (2)}$$

From equation (1) and (2) we have

$$0.711\,T_{BC} + T_{BC} = 2 \times 10^6$$

$$\therefore \underline{T_{BC} = 1168907.072\ \text{Nmm}} \quad \text{--- (3)}$$

$$\text{and} \quad \underline{T_{AB} = 831092.928\ \text{Nmm}} \quad \text{--- (4)}$$

(d) Max shear stress.

(i) For AB :- $\frac{T_{AB}}{J} = \frac{\tau_{max}}{R}$

$$\therefore \tau_{max} = \frac{831092.928 \times (60/2)}{\frac{\pi}{32} \times 60^4}$$

$$\therefore \boxed{\tau_{max} = 19.595}\ \text{N/mm}^2 \quad \text{--- (e)}$$

(ii) For BC :- $\frac{T_{BC}}{J} = \frac{\tau_{max}}{R}$

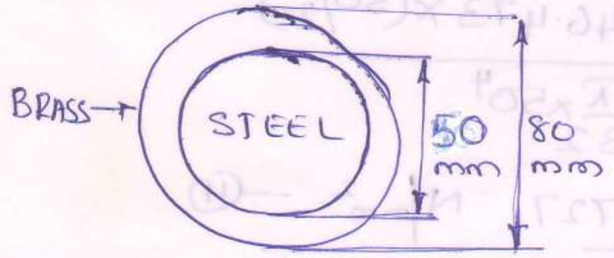
$$\tau_{max} = \frac{1168907.072 \times (60/2)}{\frac{\pi}{32}\left[60^4 - 30^4\right]}$$

$$\therefore \boxed{\tau_{max} = 29.398}\ \text{N/mm}^2 \quad \text{--- (f)}$$

Comparing (e) and (f), $\tau_{max} = 29.398\ \text{N/mm}^2$ in Hollow Section BC

(4) A brass tube of external diameter 80 mm and internal diameter 50 mm is closely fitted to a steel rod of 50 mm diameter to form a composite shaft. If a torque of 6 KNm is to be resisted by this shaft, find the maximum stresses developed in each material and the angle of twist in 2m length.
Take $G_b = 40\times10^3\ N/mm^2$ and $G_s = 80\times10^3\ N/mm^2$

Soln :-



Data :-

$d_{bo} = 80\ mm$, $d_{bi} = 50\ mm$

$d_s = 50\ mm$

$T = 6\,KNm = 6\times10^6\ N\,mm$

$L = 2m = 2\times10^3\ mm$

$G_b = 40\times10^3\ N/mm^2$

$G_s = 80\times10^3\ N/mm^2$

(a) Torsion equation, $\dfrac{T}{J} = \dfrac{\tau}{r} = \dfrac{G\theta}{L}$ — (a).

(b) Max shear stress

Net Torque, $T = T_b + T_s$ — (b)

$\therefore\ T_b + T_s = 6\times10^6\ N\,mm$ — (1)

Angle of twist, $\theta_b = \theta_s = \theta$.

$$\left(\frac{T.L}{JG}\right)_b = \left(\frac{TL}{JG}\right)_s.$$

$$\frac{T_b\times2\times10^3}{\frac{\pi}{32}\left[80^4-50^4\right]\times40\times10^3} = \frac{T_s\times2\times10^3}{\frac{\pi}{32}\times50^4\times80\times10^3}.$$

$\therefore\ \boxed{T_b = 2.7768\,T_s}$ — (2) Comparing (1) & (2)

$$\left.\begin{aligned} T_s &= 1588646.473\ N\,mm \\ T_b &= 4411353.527\ N\,mm \end{aligned}\right\} \text{— (3)}$$

(i) For brass :- $\dfrac{T_b}{J_b} = \dfrac{\tau_{max}}{R}$

$$\therefore \tau_{max} = \frac{4411353.527 \times (80/2)}{\frac{\pi}{32}\left[80^4 - 50^4\right]}$$

$$\therefore \tau_{max} = 51.781 \ N/mm^2 \quad \text{— (3)}$$

(ii) For steel :- $\dfrac{T_s}{J_s} = \dfrac{\tau_{max}}{R}$

$$\therefore \tau_{max} = \frac{1588646.473 \times (50/2)}{\frac{\pi}{32} \times 50^4}$$

$$\therefore \tau_{max} = 64.727 \ N/mm^2 \quad \text{— (4)}$$

From (3) and (4), $\tau_{max} = 64.727 \ N/mm^2$ in steel.

(c) Angle of twist :

(i) $$\theta_b = \frac{T_b L_b}{J_b G_b} = \frac{4411353.527 \times 2 \times 10^3}{\frac{\pi}{32}\left[80^4 - 50^4\right] \times 40 \times 10^3} = 0.0647 \text{ rad}$$

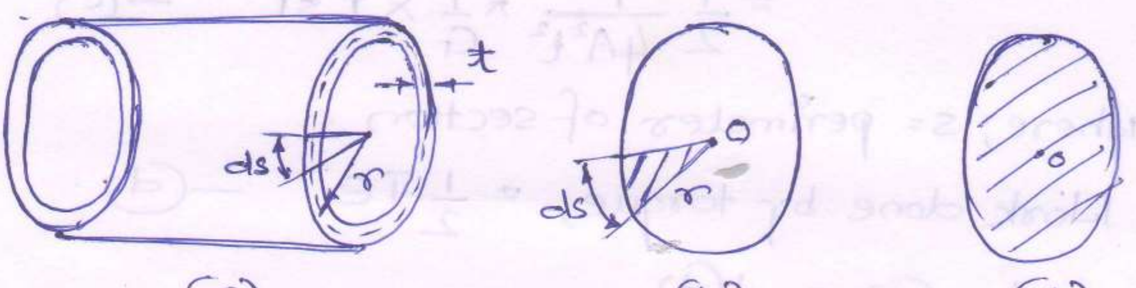
$$\therefore \boxed{\theta_b = 3.708^\circ} \quad \text{— (5)}$$

(ii) $$\theta_s = \frac{T_s L_s}{J_s G_s} = \frac{1588646.473 \times 2 \times 10^3}{\frac{\pi}{32} \times 50^4 \times 80 \times 10^3} = 0.0647 \text{ rad}$$

$$\therefore \boxed{\theta_s = 3.708^\circ} \quad \text{— (6)}$$

Hence, $\theta_b = \theta_s = \theta$

TORSION OF HOLLOW THIN WALLED SHAFTS



(a) (b) (c)

Consider a hollow circular shaft as shown in fig (a)
If the walls are thin [i.e $t < D/20$] then, the torsion equation $\frac{T}{J} = \frac{\tau}{r} = \frac{G\theta}{L}$ does not give satisfactory results.

Let, t = thickness of thin shaft of arbitrary cross section.
τ = shear stress on the element
ds = element length.
r = Radius of curvature.

Since thickness is small, τ = constant across the element.

Torque resisted, T' = shear force on element $\times r$

$$\therefore T' = \tau t ds \times r$$

$$\therefore T' = \int \tau t r \, ds \quad \text{--- (a)}$$

Torque resisted = Torque applied i.e $T' = T$

$$\therefore T = \tau t \int r \, ds \quad \text{--- (b)}$$

From fig (b), $r ds$ = Area of shaded triangle $\times 2$

$\therefore \int r ds$ = Area enclosed by median line $\times 2$ of the c/s

$$\therefore \int r ds = 2 \times A$$

$$\therefore T = \tau t \, 2A$$

$$\therefore \tau = \frac{T}{2At} \quad \text{--- (1)}$$

Angle of twist :- Strain energy = Work done.

Shearing strain energy $= \frac{1}{2}\frac{\tau^2}{G} \times V$

$$= \frac{1}{2}\frac{T^2}{4A^2t^2} \times \frac{1}{G} \times tsL \quad -(c)$$

where, s = perimeter of section

Work done by torque, $= \frac{1}{2}T\theta \quad -(d)$

Equating (c) and (d)

$$\frac{1}{2}\frac{T^2}{4A^2t^2} \times \frac{1}{G} \times tsL = \frac{1}{2}T\theta$$

$$\therefore \frac{\theta}{L} = \frac{Ts}{4A^2tG}.$$

$\therefore$ rotation per unit length $\boxed{\frac{\theta}{L} = \frac{\tau s}{2AG}} \quad -(2) \quad \because \tau = \frac{T}{2At}$

Case (i) :- If "r" mean radius of very thin circular section.

then, $s = 2\pi r$ and $A = \pi r^2$

$$\therefore \tau = \frac{T}{2\pi r^2 t} \quad -(3)$$

$$\frac{\theta}{L} = \frac{T}{2\pi r^2 t} \times \frac{2\pi r}{2\pi r^2 G} = \frac{T}{2\pi r^3 G t} \quad -(4)$$

Problem:

(1) A hollow circular shaft of outer diameter 100 mm and inner diameter 95 mm is subjected to a torque of 2KNm. Determine the value of maximum shear stress and angle of twist per unit length.

(a) Treating it as a hollow shaft (b) Treating it as a very thin shaft. Take modulus of rigidity as "G"

Soln:

Data:-

$d_o = 100\,mm$

$d_i = 95\,mm$

$T = 2\,kNm = 2\times10^6\,Nmm$

Soln:- (a) Treating as a hollow shaft

Torsion equation, $\frac{T}{J} = \frac{\tau}{r} = \frac{G\theta}{L}$

(i) $\tau_{max} = \frac{T}{J}R = \frac{T}{\frac{\pi}{32}[d_o^4 - d_i^4]} \times \frac{d_o}{2}$

$$= \frac{2\times10^6}{\frac{\pi}{16}[100^4 - 95^4]} \times 100.$$

$$\therefore \tau_{max} = 54.92\ N/mm^2 \quad \text{— (1)}$$

(ii) Angle of twist per unit length

$$\frac{\theta}{L} = \frac{T}{JG} = \frac{2\times10^6}{\frac{\pi}{32}[100^4 - 95^4]\times G} = \frac{1.098}{G} \quad \text{— (2)}$$

(b) Treating as thin walled

(i) $\tau_{max} = \frac{T}{2At} = \frac{T}{2\pi r^2 t} = \frac{2\times10^6}{2\pi\left(\frac{97.5}{2}\right)^2\times2.5}$

mean diameter, $d = \frac{d_o + d_i}{2} = 97.5\,mm$

thickness, $t = \frac{d_o - d_i}{2} = 2.5\,mm$

$$\therefore \tau_{max} = 53.57\ N/mm^2 \quad \text{— (3)}$$

(ii) $\frac{\theta}{L} = \frac{T}{2\pi r^3 t G} = \frac{2\times10^6}{2\pi\, 48.75^3 \times 2.5\,G} = \frac{1.099}{G}$ — (4)

Proof for:

Hollow shaft is stiffer than solid shaft.

Torsional equation is given by

$$\frac{T}{J} = \frac{\tau}{r} = \frac{G\theta}{L} \quad \text{— (a)}$$

Torsional rigidity/stiffness $= \frac{T}{\theta} = \frac{JG}{L}$

$$\therefore \frac{\text{Torsional stiffness of hollow shaft}}{\text{Torsional stiffness of solid shaft}} = \frac{(T/\theta)_H}{(T/\theta)_S} = \frac{\left(\frac{JG}{L}\right)_H}{\left(\frac{JG}{L}\right)_S} \quad \text{— (b)}$$

$$\therefore \left(\frac{JG}{L}\right)_H = \frac{\pi}{32}\left[D_o^4 - D_i^4\right] \times \left\{\frac{G}{L}\right\}_H \quad \text{—}\otimes$$

$$\left(\frac{JG}{L}\right)_S = \frac{\pi}{32} D^4 \times \left(\frac{G}{L}\right)_S \quad \text{—}\otimes$$

Since hollow and solid shaft material and length are same $L_H = L_S$ & $G_H = G_S$

$$\therefore \text{Eq}^n \text{ (b) becomes, } \frac{(T/\theta)_H}{(T/\theta)_S} = \frac{\frac{\pi}{32}(D_o^4 - D_i^4)}{\frac{\pi}{32} D^4} = \frac{D_o^4 - D_i^4}{D^4} \quad \text{— (c)}$$

Assuming diameter of solid shaft as mean diameter of hollow shaft

$$\therefore D = \frac{D_o + D_i}{2}$$

Hence, $D < D_o$

$$\therefore \text{Eq}^n \text{ (c), } \frac{(T/\theta)_H}{(T/\theta)_S} = \frac{D_o^4}{D^4} - \frac{D_i^4}{D^4} \quad \text{— (d)}$$

where $\frac{D_o}{D} > 1$ & $\frac{D_i}{D} < 1$

$$\therefore \frac{(T/\theta)_H}{(T/\theta)_S} > 1$$

$$\therefore (T/\theta)_H > (T/\theta)_S$$

Hence proved that torsional stiffness of hollow shaft is greater than torsional stiffness of solid shaft.

Proof for Hollow shaft is stronger than solid shaft.

Soln: Torsion equation.

$$\frac{T}{J} = \frac{\tau}{r} = \frac{G\theta}{L} \quad \text{—(a)}$$

(1) Torsional strength $= \frac{T}{\tau}$

$$\therefore \frac{\text{Torsional strength of hollow shaft}}{\text{Torsional strength of solid shaft}} = \frac{(T/\tau)_H}{(T/\tau)_S} = \frac{(J/r)_H}{(J/r)_S} \quad \text{—(b)}$$

$$\left.\begin{aligned} \therefore \left(\frac{J}{r}\right)_H &= \frac{\pi}{32}\left[D_o^4 - D_i^4\right] \Big/ \frac{D_o}{2} = \frac{\pi}{16}\frac{\left[D_o^4 - D_i^4\right]}{D_o} \\ \left(\frac{J}{r}\right)_S &= \frac{\pi}{32} D^4 \Big/ \frac{D}{2} = \frac{\pi}{16} D^3 \end{aligned}\right\} \text{—(i) Put in (b).}$$

$$\therefore \frac{(T/\tau)_H}{(T/\tau)_S} = \frac{\left(D_o^4 - D_i^4\right)}{D^3 D_o} = \left[\frac{D_o^3}{D^3} - \frac{D_i^4}{D^3 D_o}\right] \quad \text{—(c)}$$

(1) If diameter of solid shaft is assumed to be mean diameter of hollow shaft.

i.e $D = \frac{D_o + D_i}{2}$ then $\underline{\underline{D < D_o}}$

(2) Also $\frac{D_i}{D_o} < 1$,

Thus in eqn (c), $\frac{D_o^3}{D^3}$ will be greater than 1 $\left[\because D < D_o\right]$

and $\frac{D_i^4}{D^3 D_o} < 1$

$$\therefore \frac{(T/\tau)_H}{(T/\tau)_S} > 1 \qquad \therefore \left(\frac{T}{\tau}\right)_H > \left(\frac{T}{\tau}\right)_S$$

$\therefore$ Torsional strength of Hollow shaft $>$ Torsional strength of solid shaft